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TESTS OF FUNCTIONAL MEASUREMENT THEORY FOR MULTIPLICATIVE MODELS

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Abstract

This paper reports tests of functional measurement theory for multiplicative models of information integration, and shows that ratio-scale measures of mental magnitude can be computed.

Functional measurement is a theory of measurement of mental magnitude and of quantitative information involved in the generation of mental magnitude (Anderson 1981 1982 1996 2001). For example, when one lifts an object while looking at it, the heaviness of the object results from the integration of muscular quantitative information about object weight and visual quantitative information about object size (Anderson 1970). Functional measurement provides measures such as those of this heaviness and of this muscular and visual quantitative information.

		Variable Y					
		y_1	y_2	…	y_j	…	y_J
	x_1	R_{11}	R_{12}	…	R_{1j}	…	R_{1J}
	x_2	R_{21}	R_{22}	…	R_{2j}	…	R_{2J}
	…	…	…	…	…	…	…
Variable X	x_i	R_{i1}	R_{i2}	…	R_{ij}	…	R_{iJ}
	…	…	…	…	…	…	…
	x_I	R_{I1}	R_{I2}	…	R_{Ij}	…	R_{IJ}

Table 1. Representation of a factorial table reporting mean self-estimates, R_{ij}, of mental magnitude produced by the ordered values x_i and y_j of two independent variables X and Y, respectively, with i the integers from 1 to I and j the integers from 1 to J.

To obtain measures of quantitative information involved in the generation of mental magnitude, the joint use of a factorial experimental design and of a method of self-estimation involving a linear response function is required. In the simplest factorial design, I values x_i of some variable X increasing with i, and J values y_j of some variable Y increasing with j, are predefined, with i and j being integers in the intervals (1, I) and (1, J), respectively. Experimental stimuli are constructed each with a different combination of x_i and y_j. For example, stimuli could be cylinders with different combinations of volume and weight. In each stimulus, subjects self-estimate the magnitude ρ_{ij} of some mental attribute produced by x_i and y_j. For example, subjects could rate the magnitude ρ_{ij} of the heaviness of a lifted cylinder. Table 1 represents a factorial table reporting mean self-estimates R_{ij} of ρ_{ij}.

For sake of simplicity, here we consider only the cases when the number of observations in the cells of the factorial table is as large as to make standard errors of means negligible.

Functional measurement assumes that the response function is

$$R_{ij} = c_0 + c_1\, \rho_{ij} \qquad [1]$$

with c_0 and c_1 unknown constants. Thus, R_{ij} is assumed to be a measure of ρ_{ij} on an interval scale.

At a neural level, let the quantitative information about x_i and y_j be ξ_i and ψ_j, respectively. This information must be integrated to produce ρ_{ij}. One common rule of integration is represented by the multiplicative model

$$\rho_{ij} = \xi_i \; \psi_j \,. \qquad [2]$$

The response function and the multiplicative model jointly determine the functional measures of ξ_i and of ψ_j. These measures are derived as follows (Anderson 1981 1982).

The means of R_{ij} for each Row i and for each Column j of Table 1 are

$$\overline{R}_i = \sum_{j=1}^{J} \frac{R_{ij}}{J} \text{ and } \overline{R}_j = \sum_{i=1}^{I} \frac{R_{ij}}{I}, \text{ respectively.}$$

Equations 1 and 2 imply that

$$\overline{R}_i = \sum_{j=1}^{J} \frac{c_0 + c_1 \, \xi_i \; \psi_j}{J} \text{ and } \overline{R}_j = \sum_{i=1}^{I} \frac{c_0 + c_1 \, \xi_i \; \psi_j}{I}, \text{ respectively.}$$

Since each ξ_i of a given row is the same for each column, and each ψ_j of a given column is the same for each row, these means are

$$\overline{R}_i = c_0 + c' \, \xi_i \qquad [3]$$

and

$$\overline{R}_j = c_0 + c'' \, \psi_j \qquad [4]$$

with $c' = \sum_{j=1}^{J} \frac{c_1 \, \psi_j}{J}$ and $c'' = \sum_{i=1}^{I} \frac{c_1 \, \xi_i}{I}$ constants specific of the factorial design being used.

Thus, $\overline{R}_i$ and $\overline{R}_j$ are interval-scale measures of ξ_i and ψ_j, respectively.

Tests

By putting Equations 1–4 together and rearranging one obtains the linear relations

$$R_{ij} = c_0 - k \, c_0 + k \, \overline{R}_i \quad \text{with} \quad k = \frac{c_1}{c' \, c''} (\overline{R}_j - c_0) \qquad [5]$$

and

$$R_{ij} = c_0 - k' \, c_0 + k' \, \overline{R}_j \quad \text{with} \quad k' = \frac{c_1}{c' \, c''} (\overline{R}_i - c_0) \,. \qquad [6]$$

Several empirical tests have been made which confirm different predictions from Equations 5 and 6 (Anderson, 1981 1982 1991 1996). The following are additional tests of these equations.

Test 1. If the intercept $c_0 - k\, c_0$ in Equation 5 increases with $\overline{R}_j$ then c_0 is negative, and if this intercept decreases as $\overline{R}_j$ increases then c_0 is positive. Equation 6 involves a similar test.

Test 2. Functional measurement theory for multiplicative models provides ratio-scale measures of ξ_i and ψ_j (Anderson, 1982, pp. 82-83). These measures may be obtained as follows. For each Row i and each Column j consider, respectively, the differences with minimum relative error

$$D_i = \left| R_{i1} - R_{iJ} \right| \qquad [7]$$

and

$$D_j = \left| R_{1j} - R_{Ij} \right| . \qquad [8]$$

By putting Equations 1, 2, and 7 together and rearranging one obtains

$$D_i = u\, \xi_i \qquad [9]$$

and by putting Equations 1, 2, and 8 together and rearranging one obtains

$$D_j = v\, \psi_j \qquad [10]$$

with $u = c_1 \left| \psi_1 - \psi_J \right|$ and $v = c_1 \left| \xi_1 - \xi_I \right|$ constants specific of the factorial design being used.

Thus, D_i and D_j are functional ratio-scale measures of ξ_i of ψ_j, respectively.

Finally, by putting Equations 1, 2, 9, and 10 together and rearranging one obtains

$$R_{ij} = c_0 + \frac{c_1}{u\, v} D_i\, D_j \, . \qquad [11]$$

By assumption c_0 is a constant. Accordingly, Equation 11 predicts that c_0 is invariant with D_j when R_{ij} is plotted as a function of D_i, and is invariant with D_i when R_{ij} is plotted as a function of D_j.

Tests 1 and 2 applied on Anderson and Butzin's results

Anderson and Butzin (1974) tested empirically the model that the performance attributed by a subject to an individual is equal to the product of motivation and ability attributed by the subject to the same individual. On a 20-cm graphic bar, labeled High and Low at the ends, twenty subjects rated the performance of applicants to graduate school. A 4 (motivation) × 4 (ability) factorial design was used. Each level of motivation or of ability was stated to the subject as either low (L), slightly below average (M^-), slightly above average (M^+), or high (H). Subjects rated performance of applicants for each combination of levels of motivation and ability.

Table 2 reports the mean ratings of performance, R_{ij}, derived from Anderson and Butzin's (1974) Figure 2. The interval-scale measures $\overline{R}_i$ and $\overline{R}_j$, and the ratio-scale measures D_i and D_j, calculated on these mean ratings are also reported. In Figure 1, R_{ij} is plotted as a function of the interval-scale measure $\overline{R}_i$ (left) and of the ratio-scale measure D_i (right) of motivation, with the parameters being the interval-scale measure $\overline{R}_j$ and the ratio-scale measure D_j of ability, respectively. For each parameter value, a straight line obtained by least squares fitting is depicted.

		Ability					
		y_1 (L)	y_2 (M^-)	y_3 (M^+)	y_4 (H)	$\overline{R}_i$	D_i
Motivation	x_1 (L)	3.3	5.0	7.2	9.0	6.1	5.7
	x_2 (M^-)	4.5	7.4	9.4	11.3	8.3	6.6
	x_3 (M^+)	7.6	9.9	12.4	14.5	11.1	6.9
	x_4 (H)	9.0	11.6	14.7	17.5	13.2	8.5
	$\overline{R}_j$	6.1	8.5	10.9	13.1		
	D_j	5.7	6.6	7.5	8.5		

Table 2. Mean ratings of attributed performance obtained by Anderson and Butzin (1974, Figure 2) for each combination of low (H), slightly below average (M^-), slightly above average (M^+), and high (H) levels of attributed motivation and of attributed ability.

In Figure 1, in the left diagram, the intercept of fitted lines increases with $\overline{R}_j$. Consequently, Test 1 predicts that c_0 is negative. Test 2 predicts that c_0 is invariant with D_j. The results in the right diagram show that these predictions are confirmed: the intercept c_0 of the fitted lines is negative and essentially invariant with D_j, in agreement with the assumption of Equation 1 that c_0 is a constant. When R_{ij} is plotted as a function of $\overline{R}_j$ and of D_j, c_0 is negative and essentially invariant with D_i.

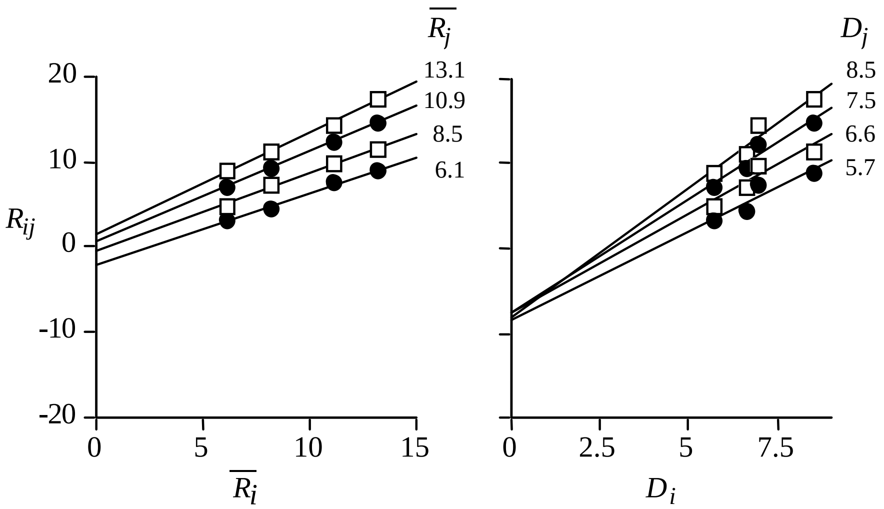

Figure 1. Mean ratings of performance (R_{ij}, reported in Table 2) as a function of an interval-scale measure $\overline{R}_i$ (left) and of a ratio-scale measure D_i (right) of motivation. The parameters are the interval-scale measures $\overline{R}_j$ and ratio-scale measures D_j of ability, respectively.

One important feature of functional measurement theory is that it allows for ratio-scale measurement of mental magnitude. In fact, Equation 1 may be rewritten as

$$R_{ij} - c_0 = c_1 \, \rho_{ij} .$$

The quantity $R_{ij} - c_0$ is a ratio-scale measure of ρ_{ij}. Since we determine R_{ij} empirically and estimate c_0 by some fitting procedure, we can obtain ratio-scale measures of mental magnitude.

Cases when $c_0 = 0$

The constant c_0 in Equation 1 depends on the specific factorial design and on the specific procedure for the method of self-estimation being used. The design and the rating procedure used by Anderson and Butzin (1974) produced a $c_0 \neq 0$. It may be that the design and the self-estimation procedure produce a $c_0 = 0$. In this case Equation 1 reduces to

$$R_{ij} = c_1 \, \rho_{ij} \, .$$

That is, in this case, subject's self-estimates are direct ratio-scale measure of ρ_{ij}.

When $c_0 = 0$, each of Equations 5 and 6 reduces to

$$R_{ij} = \frac{c_1}{c' \, c''} \, \overline{R}_i \, \overline{R}_j \qquad [12]$$

and Equation 11 reduces to

$$R_{ij} = \frac{c_1}{u \, v} D_i \, D_j \, . \qquad [13]$$

Thus, when $c_0 = 0$, the factorial graphs implied by Equations 12 and 13 are each a fan of straight lines with a common origin equal to $c_0 = 0$.

Shanteau and Anderson (1972) used a factorial design and a rating procedure that produced a $c_0 = 0$. These authors tested the model that, in making a decision, the judged worth of an added piece of probabilistic information is equal to the product of this added piece of probabilistic information and the amount of prior probabilistic information relevant for the decision. A 4 (added information) × 5 (prior information) factorial design was used. The levels of added information were the probabilities 1/6, 3/6, 5/6, and 6/6. The levels of prior information were the probabilities 0.5, 0.6, 0.7, 0.8, and 0.9. On a 50-cm graphic bar, thirty-two subjects rated worth of added information for each combination of levels of added and prior information (each single rating was multiplied by 2).

		Prior Information						
		y_1 (0.9)	y_2 (0.8)	y_3 (0.7)	y_4 (0.6)	y_5 (0.5)	$\overline{R}_i$	D_i
Added Information	x_1 (1/6)	3.6	5.9	8.8	9.3	11.1	7.7	7.5
	x_2 (3/6)	11.1	17.9	22.3	27.6	33.6	22.5	22.5
	x_3 (5/6)	18.5	28.1	36.0	45.0	54.2	36.4	35.7
	x_4 (6/6)	23.3	33.0	42.8	52.7	65.1	43.4	41.8
	$\overline{R}_j$	14.1	21.2	27.5	33.7	41.0		
	D_j	19.7	27.1	34.0	43.4	54.0		

Table 3. Mean ratings of worth of added probabilistic information obtained by Shanteau and Anderson (1972, Figures 1 and 3) for each combination of levels of added and prior probabilistic information. This information is expressed in terms of the probabilities reported within parentheses.

Table 3 reports mean ratings R_{ij} of worth of added information obtained by Shanteau and Anderson (1972, Figures 1 and 3) excluding the ratings of 10 discrepant subjects. It also reports the interval-scale measures $\overline{R}_i$ and $\overline{R}_j$, and the ratio-scale measures D_i and D_j, calculated on these mean ratings.

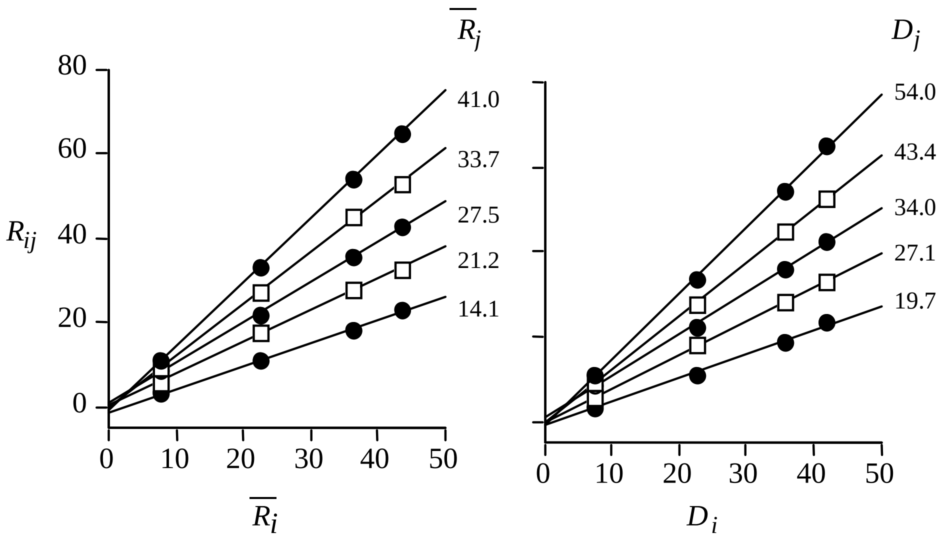

Figure 2. Mean ratings of worth of added information (R_{ij}, reported in Table 3) as a function of an interval-scale measure $\overline{R}_i$ (left) and of a ratio-scale measure D_i (right) of added information. The parameters are the interval-scale measures $\overline{R}_j$ and ratio-scale measures D_j of prior information, respectively.

In Figure 2, R_{ij} is plotted as a function of the interval-scale measure $\overline{R}_i$ (left) and of the ratio-scale measure D_i (right) of added information. The parameters are the interval-scale measure $\overline{R}_j$ and the ratio-scale measure D_j of prior information, respectively. For each parameter value, a straight line obtained by least squares fitting is depicted.

In Figure 2, the intercept of fitted lines in the left diagram is essentially $c_0 = 0$. Accordingly, also the intercept of fitted lines in the right diagram is essentially $c_0 = 0$. Similar results are obtained when R_{ij} is plotted of a function of $\overline{R}_j$ and of D_j.

In conclusion, the present tests confirm functional measurement theory for multiplicative models.

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